Simple Models for Scale Dependent Spectral Dimension

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Simple models for scale dependent spectral dimension

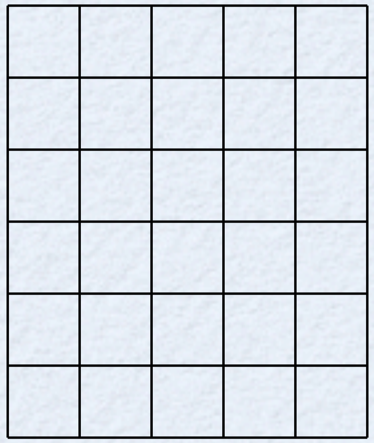
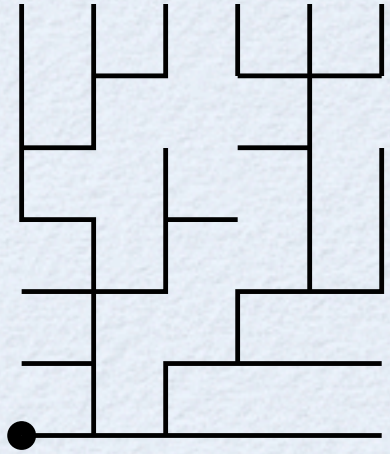
1. Characterizing large scale structure of graphs
2. A curious result from quantum gravity
3. A definition
4. An easy example
5. An example using random graph ensembles
6. Conclusions

1. Characterizing Large Scale Structure

Hausdorff dimension d_H -- we assume ∞ graphs

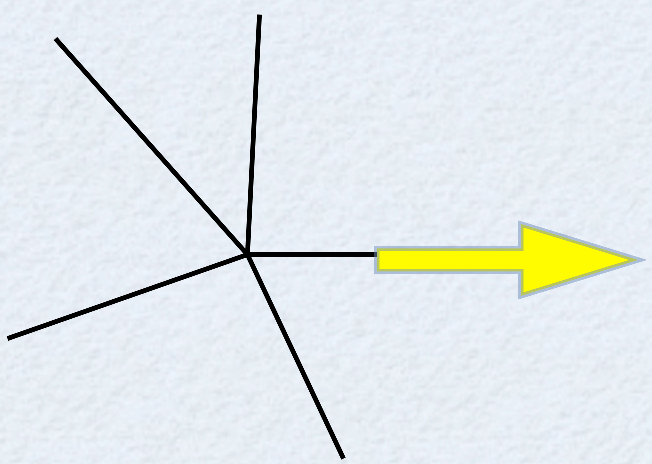
1. Choose a point r_0
2. Find all points $B_R(r_0)$ within graph distance R of r_0
3. $|\ B_R(r_0)\ | \sim R^{d_H}$ as $R \rightarrow \infty$, independent of r_0

d_H is blind to some sorts of connectivity eg in the Euclidean metric

$d_H = 2$ for Z^2  & Spanning Tree 

Spectral dimension d_s

1. Choose a point r_0
2. Random walker leaves r_0 at time 0 and returns at time t with probability $q_G(t;r_0)$

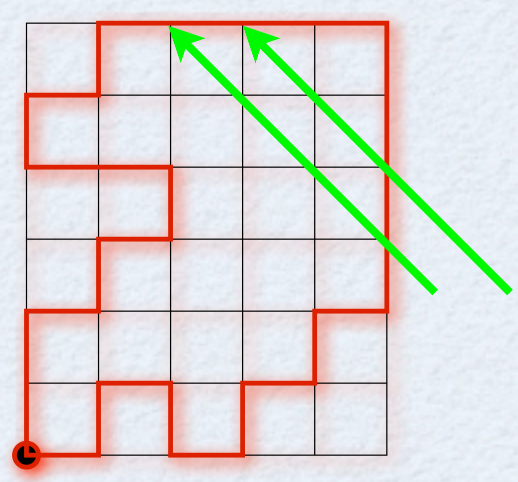


prob = σ^{-1}

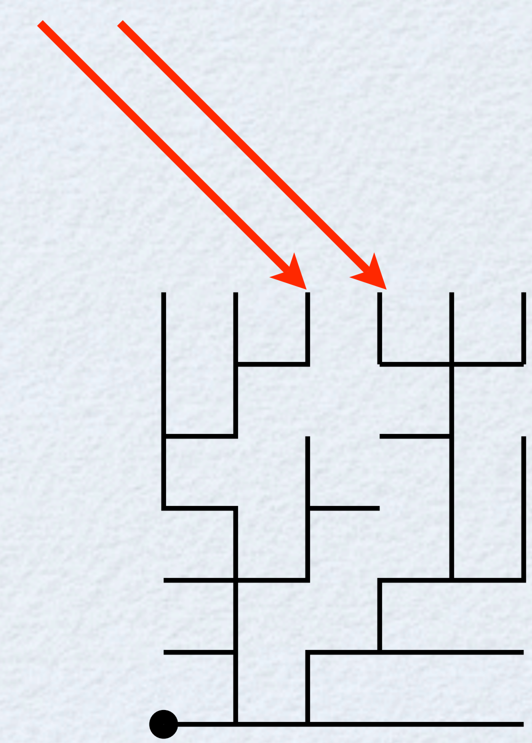
$q_G(t;r_0) \sim t^{-d_s/2}$ as $t \to \infty$

Random walk sees connectivity:

$d_s = 2$ for Z^2



but $16/13$ for



2. A curious result from quantum gravity

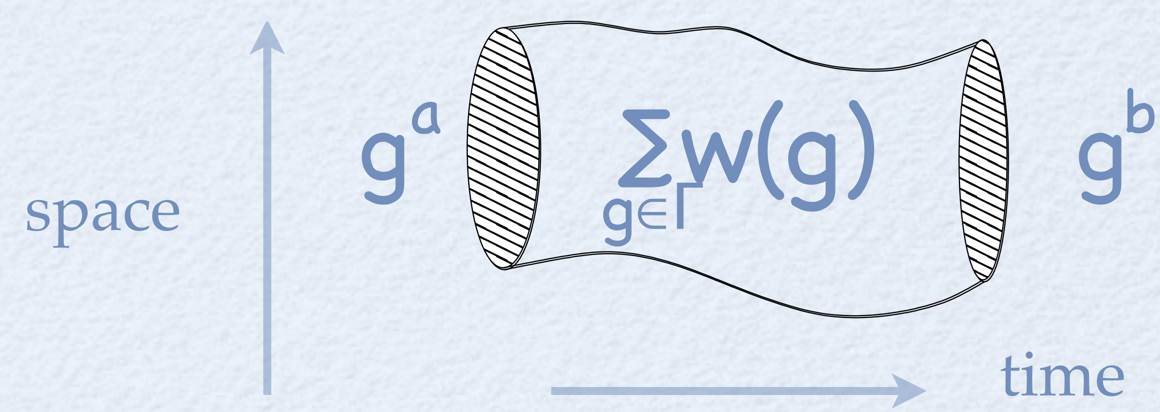
Gravity's dynamical degree of freedom is the metric $g_{\mu\nu}(x,t)$

Classically $g_{\mu\nu}(x,t)$ obeys Einstein's equations:

$$g_{\mu\nu}(x,0) \longrightarrow g_{\mu\nu}(x,t)$$

Quantum mechanics is different:

$$\langle g^b(x),\ t=T \mid g^a(x),\ t=0 \rangle \sim$$



Probability amplitude for evolution from g^a to g^b

How are Γ and w defined ?

Several approaches; some non-stringy ones ....

1. Continuum field theoretical -- exact RG looking for non-Gaussian fixed points where QG non-perturbatively renormalizable (asymptotic safety)

hep-th/0508202

2. Discretized -- Causal Dynamical Triangulations Γ is a set of graphs and we look for a critical point (or line) where a continuum limit can be taken to recover QG

hep-th/0505113

Curiosity is that both indicate $d_s=4$ at large distance scales

but $d_s=2$ at small distance scales

3. Defining scale dependent d_s

Convenient to use generating fn

$$Q_G(x,L) = 1+\sum_{t=2}^{\infty} q_G(t;r_0)\,(1-x)^{t/2}$$

$$\sim x^{-1+d_s/2} \text{ as } x \to 0$$

Now define (if the limit exists),

$$\tilde{Q}(\xi,\lambda) = \lim_{a \to 0} a^{1/2}\, Q(\, a\xi,\, (\lambda/a)^{\Delta})$$

And split the sum $\sum_{t=2}^{\infty}\ldots\ldots = \sum_{t=2}^{T}\ldots\ldots + \sum_{t=T}^{\infty}\ldots\ldots$

Make suitable choices of T and bound one of the sums

Choose $$T = \frac{1}{a\xi \log\left(1 + (\xi\lambda)^{-1}\right)}$$

$$\tilde{Q}(\xi,\lambda)\,(1 - e^{-\xi\lambda}) < \lim_{a \to 0} a^{1/2} \sum_{t=2}^{T} q_G(t)\,(1-a\xi)^{t/2} < \tilde{Q}(\xi,\lambda)$$

$\tilde{Q}(\xi \to \infty,\lambda)$ describes walks of continuum duration $< \lambda$

Choose $$T = \frac{\log(1 + \xi\lambda)}{a\xi}$$

$$\tilde{Q}(\xi,\lambda) - e a^{1/2}\sqrt{T} < \lim_{a \to 0} a^{1/2} \sum_{t=T}^{\infty} q_G(t)\,(1-a\xi)^{t/2} < \tilde{Q}(\xi,\lambda)$$

$\tilde{Q}(\xi \to 0,\lambda)$ describes walks of continuum duration $> \lambda$

So define spectral dimension at long distances by

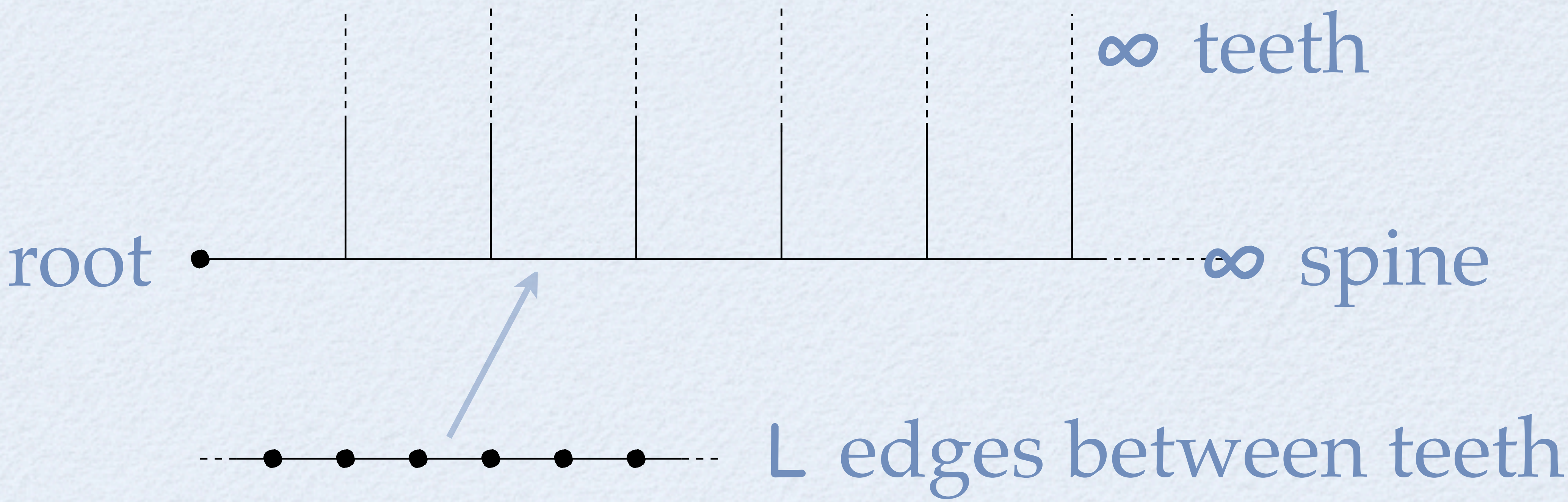
$$1 + d_s/2 = \lim_{\xi \to 0} \frac{\log \tilde{Q}(\xi,\lambda)}{\log \xi}$$

and at short distances by

$$1 + d_s/2 = \lim_{\xi \to \infty} \frac{\log \tilde{Q}(\xi,\lambda)}{\log \xi}$$

The existence and ordering of the limits are crucial

4. An easy example



$Q(x,L)$ can be calculated exactly

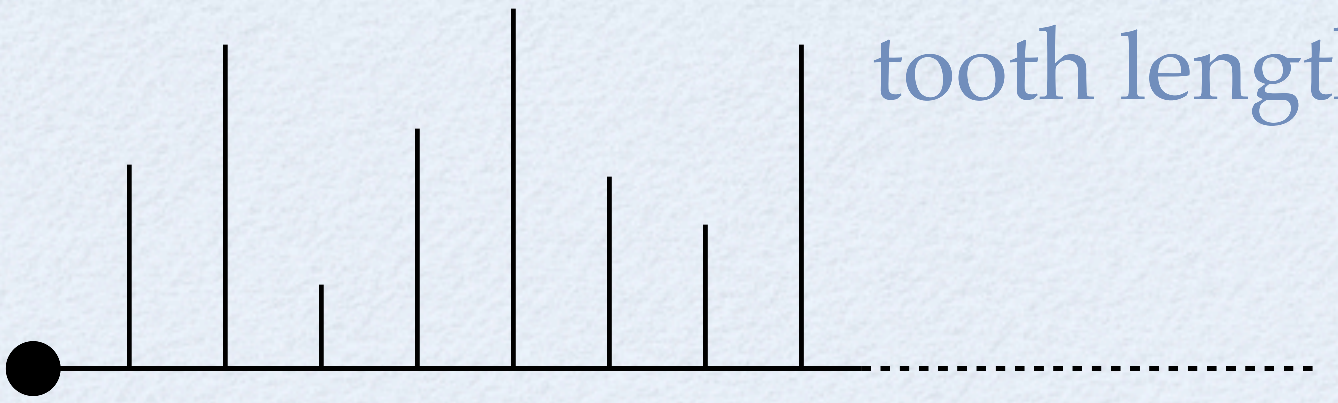
$$\tilde{Q}(\xi,\lambda) = \lim_{a\to 0} a^{1/2}\, Q(\, a\xi,\, (\lambda/a)^{1/2})$$

$$= \frac{2}{\xi^{1/2}\,(\,5 + 4\,\coth(\lambda\xi)^{1/2}\,)^{1/2}}$$

$\xi \to 0,\ d_s=3/2$

$\xi \to \infty,\ d_s=1$

5. Random Ensembles after hep-th/0509191



tooth length i.i.d. $\pi(0) = 1 - 1/L$,

$\pi(n > 0) = C\, n^{-\alpha}/L$

∞ spine

Interesting range is $1 < \alpha \leq 2$; for L fixed

$$d_H = 3 - \alpha,$$
$$d_S = 2 - \alpha/2$$

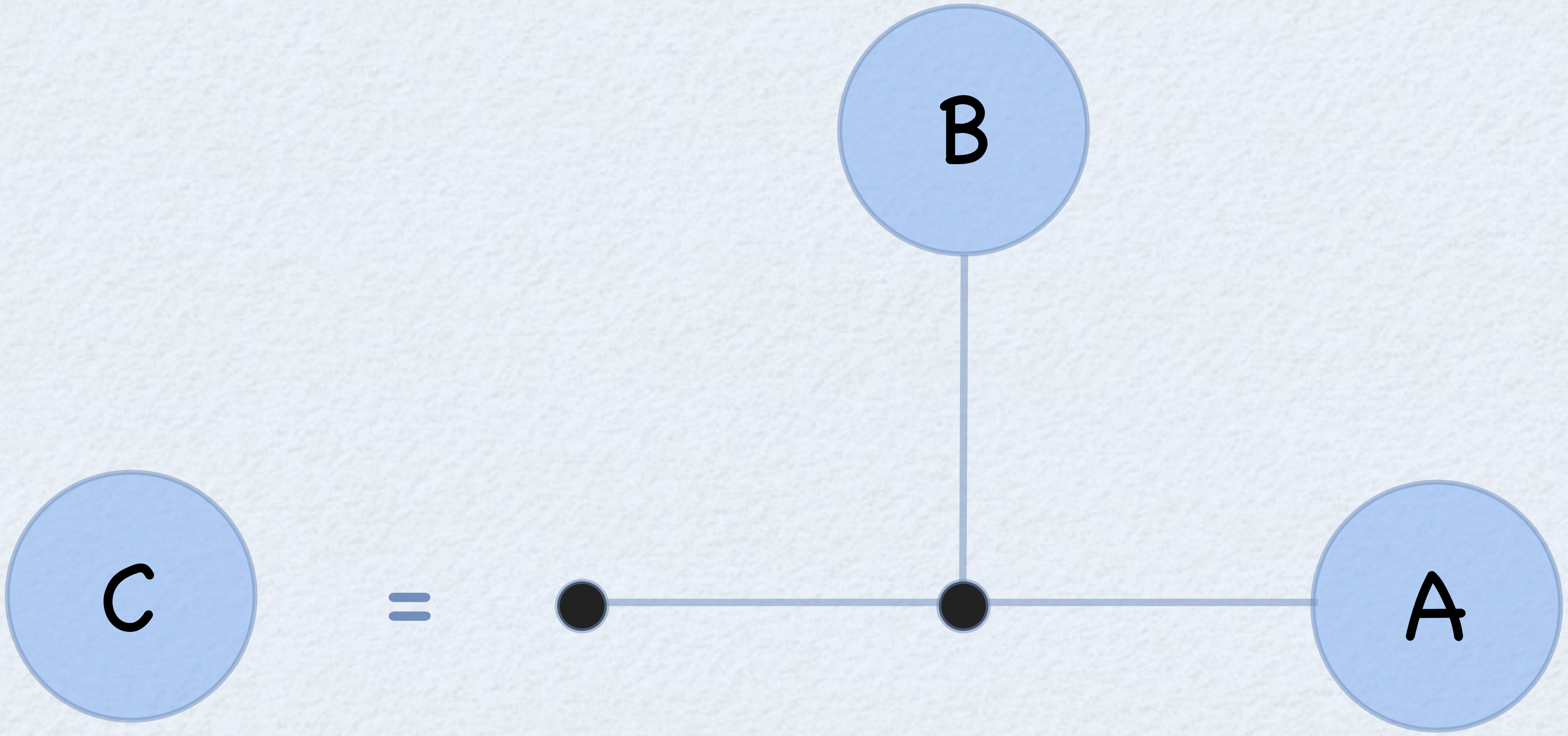
True both for expectation values and a.s. for a single comb. The right scaling limit is

$$\langle \tilde{Q}(\xi,\lambda) \rangle = \lim_{a \to 0} a^{1/2} \quad \langle Q(\, a\xi,\, (\lambda/a)^{1-\alpha/2}) \rangle$$

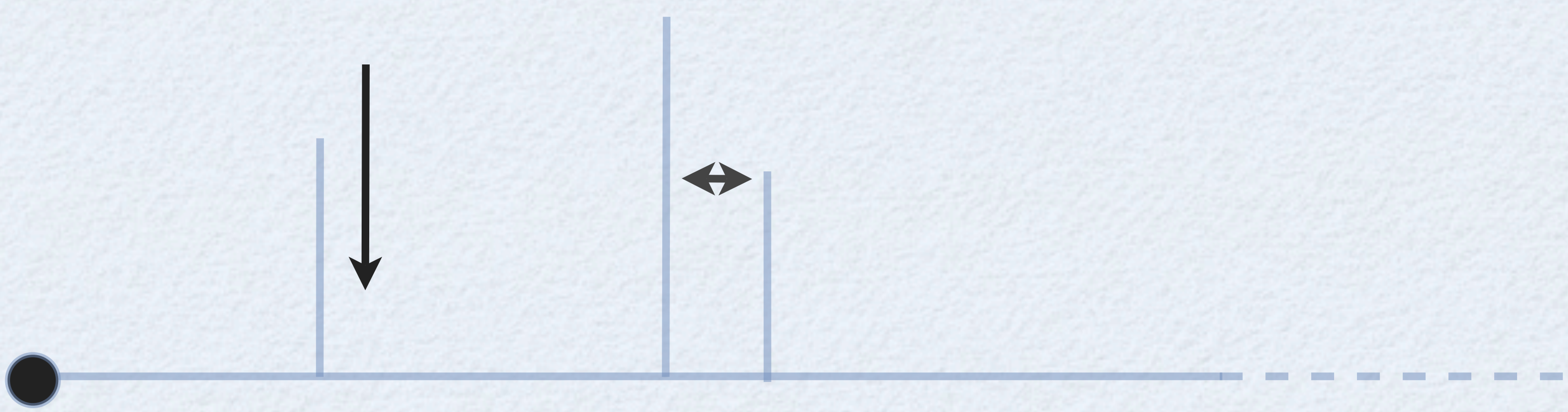
First return probabilities related by

$$P_C = \frac{1-x}{3-P_A-P_B}$$

$$Q_G(x) = \frac{1}{1 - P_G(x)}$$



Jensen's inequality gives lower bound



Graph moves give upper bound

Altogether we get

$$\frac{c}{\xi^{1/2}\,(\,1 + b\,(\lambda\xi)^{\alpha/2-1}\,)^{1/2}} < Q(\xi,\lambda) < \frac{1}{\xi^{1/2}}\,F(\lambda\xi)$$

with $F(v) \sim \begin{cases} \text{const}, & v \to \infty \\ v^{\alpha/4}, & v \to 0 \end{cases}$

$\xi \to 0,\ d_s = 2 - \alpha/2$

$\xi \to \infty,\ d_s = 1$

6. Conclusions

1. Spectral dimension can be defined on different scales
2. Order of limits is very important
3. We have considered only "kinematics" -- the probabilities $\pi(n)$ were put in by hand
4. Interesting to find solvable models where the affect appears dynamically